Acta of Turin Polytechnic University in Tashkent

Volume 9
Issue 2 *The Annual Colloquia, Seminars and Exhibition of the Turin Polytechnic University in Tashkent, 2019*

Article 3

6-1-2019

Some Properties of Removable Singular Sets of Subharmonic Functions

Sh.Sh. Shopulatov
Institute of Mathematics, Uzbekistan Academy of Sciences, shomurod_shopulatov@mail.ru

ACTA TTPU

I. FUNDAMENTAL SCIENCE

SOME PROPERTIES OF REMOVABLE SINGULAR SETS OF SUBHARMONIC FUNCTIONS

Sh.Sh.Shopulatov

Institute of Mathematics, Uzbekistan Academy of Sciences

shomurod_shopulatov@mail.ru

Abstract

The purpose of this paper is to show some properties of $\overline{S},\underline{S}$ - singular removable sets to subharmonic functions.

Keywords: subharmonic functions, removable singular sets, generalised Laplace operator.

In the theory of subharmonic functions the expression of Blaschke [1]

$$\frac{\mathfrak{M}_u\left(x^0,r\right)-u\left(x^0\right)}{\frac{2}{n}\eth r^2}, \tag{1}$$

where $\mathfrak{M}_u\left(x^0,r\right)$ denotes the integral mean of a given function u on the sphere with radius r and center x^0, plays an analogous role to that of the expression

$$\frac{u(x-h)-2u(x)+u(x+h)}{h^2}$$

for convex functions. For instance, in order that an upper semi-continuous function $u:D\to[-\infty,\infty),$ should be subharmonic in domain $D\subset\mathbb{R}^n, n\geq 2$, it is necessary and sufficient that the upper limit of (1) as $r\to 0$ should be non-negative at any point x^0 in D [2]. The same theorem [3] holds for the operator introduced by Privalov

$$\frac{\mathfrak{N}_u\left(x^0,r\right)-u\left(x^0\right)}{\frac{2}{n+2}\eth r^2}, \tag{2}$$

where $\mathfrak{N}_u\left(x^0,r\right)$ denotes the integral mean of u over the ball with radius r and center x^0. S.Saks [4] proved that for any subharmonic function u the limits (1) and (2), as $r\to 0$, exist, and are equal, almost everywhere. This may be considered as an extension of the known fact that any convex function has almost everywhere the second generalised derivative of Schwarz (second symmetrical derivative). I.Privalov showed that the condition of subharmonicity can be weakened. He got more deeper result with exceptional set E [5]. Pokrovskii [7] showed that the well-known Blaschke-Privalov local condition for a subharmonic function $u(x)$ in a domain $D\subset\mathbb{R}^n$, in terms of ball means, can be replaced on some subsets of D by another a priori more weak local conditions of the same type. In [8] we defined the singular sets $\overline{S},\underline{S}$ and proved the generalisation of Privalov's theorem.

The purpose of this paper is to show some properties of $\overline{S},\underline{S}$ - singular removable sets to subharmonic functions.

Note that, subharmonic function is summable, i.e. $u(x)\in L^1_{loc}(D)$ and its Laplace operator Ä$u\geq 0$ in the generalisation sense. Now we give the construction of the upper and the lower generalised Laplace operators [6]. Let $u(x)$ be an upper semi-continuous function in a domain $D \subset \mathbb{R}^n$ and $u(x) \not\equiv -\infty$. We define

$\mathfrak{M}(x^0, r) = \frac{1}{r^{n-1}\sigma_n} \int\limits_{S(x^0,r)} u(x)\, d\sigma$ as the mean u over the sphere $S(x^0, r)$. If $u(x)$ nonsummable on the $S(x^0, r)$ we put $\mathfrak{M}_u(x^0, r) := -\infty$. For the point $x^0 \in D \setminus u_{-\infty}$, where $u_{-\infty} := \{x \in D : u(x) = -\infty\}$, we define upper $\overline{\Delta} u(x^0)$ and lower $\underline{\Delta} u(x^0)$ generalised Laplace operators of a function u at the point x^0, constructed by the mean over the areas of spheres with following equalities:

$$\overline{\Delta} u(x^0) := 2n \cdot \overline{\lim_{r \to +0}} \frac{\mathfrak{M}_u(x^0, r) - u(x^0)}{r^2}, \quad (3)$$

$$\underline{\Delta} u(x^0) := 2n \cdot \underline{\lim_{r \to +0}} \frac{\mathfrak{M}_u(x^0, r) - u(x^0)}{r^2}. \quad (4)$$

Remark. This kind of construction can also be implemented by the mean of a function u on the ball with a center at the point x^0 and a radius r.

Note, that the generalised Laplace operators $\overline{\Delta} u(x)$ and $\underline{\Delta} u(x)$ can equal to $+\infty$ and $-\infty$. For example, a function $u(z)$ of a complex variable $z = x + iy$, defined as follows,

$$u(z) = \begin{cases} -\frac{1}{|x|}, \ if \ x \neq 0 \\ 0, \quad if \ z = 0 \\ -\infty, \ if \ x = 0 \ and \ y \neq 0 \end{cases},$$

is upper semi-continuous in

$$\mathbb{C}, \mathfrak{M}_u(0, r) = -\infty \ \forall r > 0,$$

$$\overline{\Delta} u(0) = 4 \cdot \overline{\lim_{r \to +0}} \frac{-\infty - 0}{r^2} = -\infty.$$

Extra properties of generalised Laplace operators can be found in [3,6,8].

Theorem (Blaschke-Privalov) [6]. If the function $u(x), u(x) \not\equiv -\infty$, is upper semi-continuous in the domain $D \subset \mathbb{R}^n$ holds following inequality

$$\overline{\Delta} u(x) \geq 0 \quad \forall x^0 \in D \setminus u_{-\infty}$$

then $u(x)$ is subharmonic in D.

I.Privalov got more deeper result with exceptional set E. Let $E \subset \mathbb{R}^n$ be a closed set.

Theorem (Privalov) [5]. If the function $u(x), u(x) \not\equiv -\infty$, is upper semi-continuous in the domain $D \subset \mathbb{R}^n$ and holds the following conditions

$$\overline{\Delta} u(x) \geq 0 \quad \forall x^0 \in [D \setminus u_{-\infty}] \setminus E, \ mesE = 0;$$

$\overline{\Delta} u(x) > -\infty \quad \forall x^0 \in E$ except a polar set $P \subset E$,

then the function $u(x)$ is subharmonic in D.

Corollary. If the function $u(x) \in C(D)$ holds following conditions

$$\underline{\Delta} u(x) \le 0 \le \overline{\Delta} u(x), \forall x^0 \in [D \setminus u_{-\infty}] \setminus E, mesE = 0$$

$\overline{\Delta} u(x) > -\infty, \underline{\Delta} u(x) < +\infty \quad \forall x^0 \in E$ except a polar set $P \subset E$,

then the function $u(x)$ is harmonic in D.

Definition. An arbitrary set E is called $\underline{S}$ (singular), if there exists
$v(x) \in sh(\mathbb{R}^n) : \underline{\Delta} v(x)\big|_E = +\infty$.

An arbitrary set E is called $\overline{S}$ (singular), if there exists
$v(x) \in sh(\mathbb{R}^n) : \overline{\Delta} v(x)\big|_E = +\infty$.

Some properties of *singular sets:*

$\underline{S}$ - set is a $\overline{S}$ - set, i.e. $\underline{S}$ - sets $\subset \overline{S}$ - sets.

Countable union of singular $\underline{S}$ - sets is singular $\underline{S}$ - set.

Finite union of $\overline{S}$ - sets is singular $\overline{S}$ - set.

Following theorem is the generalisation of Privalov theorem [5] in terms of singular sets defined above.

Theorem [8]. Let a function $u(x)$, $u(x) \not\equiv -\infty$, is upper semi-continuous in the domain $D \subset \mathbb{R}^n$ and $\overline{\Delta} u \ge 0$ in $[D \setminus u_{-\infty}] \setminus E$. Then

a) if $E \in \underline{S}$ and $\overline{\Delta} u > -\infty$ on E, excluding a polar set $P \subset E \Rightarrow u(x) \in sh(D)$.

b) if $E \in \overline{S}$ and $\underline{\Delta} u > -\infty$ on E, excluding a polar set $P \subset E \Rightarrow u(x) \in sh(D)$.

Let $E \subset \mathbb{R}^n$ be a closed set. The following theorem that connects $\underline{S}$ - set with a Lebesgue measure holds

Theorem. $E \in \underline{S} \iff mesE = 0$.

For an arbitrary set $E \subset \mathbb{R}^n$ the following theorem is true.

Theorem. $E \in \overline{S} \Rightarrow \overset{\circ}{E} = \varnothing$.

References

1. W. Blaschke, Ein Mittelwertsatz und eine kennzeichnende Eigenschaft des logarithischen Potentials, Ber. Verh. Sächs. Akad. Wiss., 68, (1916), 3-7pp.

2. T. Rado, Subharmonic functions, Ergebnisse der Mathematik, Berlin, 1937.

3. I. Privaloff, Sur les fonctions harmoniques, Rec. math., XXXII:3, (1925), 464-469 pp.

4. S. Saks, On the operators of Blaschke and Privaloff for subharmonic functions, Rec.Math.[Mat. Sbornik] N.S., 9(51) (2), (1941), 451-456 pp.

5. I. Privalov, To the definition of a subharmonic function, News of the USSR Acad. of Sci., (5), 1941, 281-284 pp, (in Russian).

6. Brelo M., Foundations of classical potential theory, 217p. Mir, Moscow, (1964), (in Russian).

7. A. Pokrovskii, Conditions for subharmonicity and subharmonic extensions of functions, Sb. Math., 208:8, 2017, 1225-1245 pp, (in Russian).

8. Sh. Shopulatov, On a weak criterion for the subharmonicity of functions in $\mathbb{R}^n$, Uzb.Math.Jour., (2) 2018, 136-141 pp.